# Struts

## Instability

So far we have been accustomed to dealing with stable equilibrium systems. However not all structural systems are stable. A simple example of a rigid vertical bar that is pinned at the base and restrained by a spring at the top,see Fig.1, will be used to illustrate this fact. The bar is assumed to be subjected to a vertical force P and the stiffness of the sping is k.

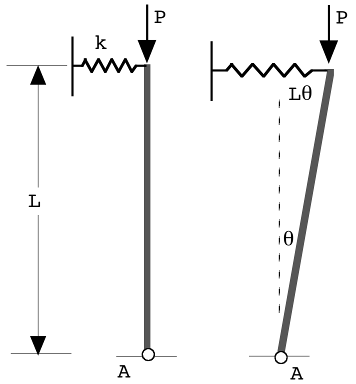



Fig.1

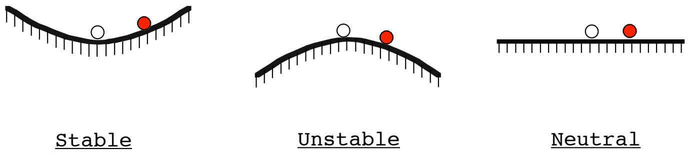



Fig.2

Now let us consider the effect of a small angular displacement θ. The extension in the spring is Lθ and the force is kLθ. By taking moments about A we get

$$kL^2\theta > PL\theta \qquad , \qquad \text{Stable equilibrium}$$

$$kL^2\theta < PL\theta \qquad , \qquad \text{Unstable equilibrium}$$

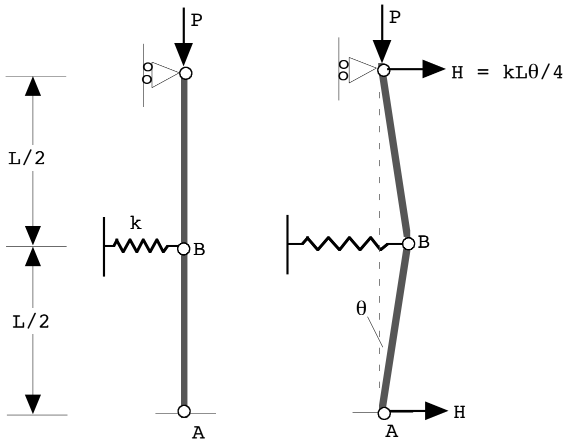
$$kL^2\theta = PL\theta \qquad , \qquad \text{Neutral equilibrium}$$

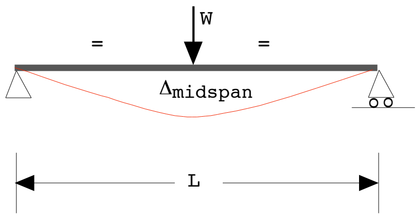
The situation is similar to a marble on a smooth surface, see Fig.2.

The condition of neutral equilibrium (the system neither stable nor unstable) reduces to

$$L\theta(P - kL) = 0$$

and so either $\theta = 0$ or $P = kL$. The first solution is trivial. However the second solution yields the *critical* or *buckling* load of the system $P_{cr} = kL$ for arbitrary small rotations θ.




$$\Delta = \frac{WL^3}{48EI} \qquad k = \frac{48EI}{L^3}$$

Fig.3

Fig.4

Let us now examine the effect of a small angular displacement θ on the system consisting of two rigid bars shown in Fig.3. Taking moments about B we get in this case

$$kL^2\theta/8 = PL\theta/2 \qquad , \qquad \text{Neutral equilibrium}$$

We obtain $P_{cr} = kL/4$. If we substitute the equivalent spring stiffness for the simply-supported beam of flexural rigidity EI shown in Fig.4, $P_{cr}$ becomes equal to $12EI/L^2$.

In both cases we observe that the citical load depends only on the spring stiffness. The strength of the material of which the bars are made does not play a part.

## Euler load for pin-ended columns

When a compression member is axially loaded failure will occur in bending long before yielding occurs. This happens because of initial imperfections i.e. the member may not be perfectly straight and its mechanical properties may not be uniform; also it may be impossible to ensure that the load is centrally (without eccentricity) applied. Fig. 5 shows the behaviour of a typical axially loaded pin-ended member.

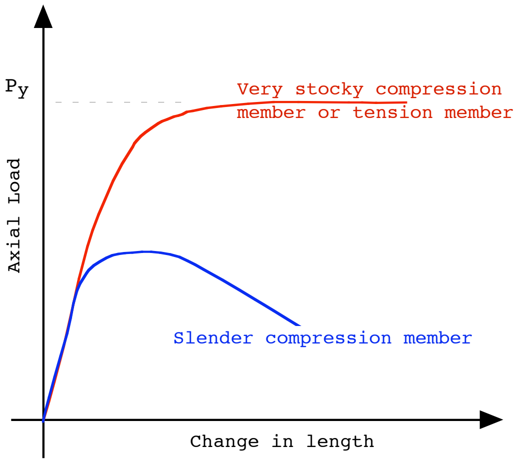



Fig.5

A very stocky member will fail by squashing. On the other hand a slender column will buckle sideways prematurely.

Let us now look at the behaviour of an ideal column. We make the following assumptions:

1. The ends are perfect pins with one of them free to move verically.

2. The material is perfectly straight and is of uniform cross-section with flexural rigidity equal to EI.

3. The material is linear elastic and no question of yielding arises under the applied load.

4. The lateral deflection, u, of the column is small in relation to its length.

The situaion is illustrated in Fig.6. For equilibrium we have

$$M + Pu = 0 \qquad [1]$$

Therefore

$$EI\frac{d^2u}{dz^2} + Pu = 0 \qquad [2]$$

or

$$\frac{d^2u}{dz^2} + k^2u = 0 \qquad [3]$$

where $k^2 = P/EI$

Equation [3] has a solution $u = \delta \sin(kz)$ [4]

Applying the boundary condition $u = 0$ @ $z = L$ we get

$$\text{either } \delta = 0 \text{ or } \sin(kL) = 0 \qquad [5]$$

For a non-trivial solution $\sin(kL) = 0$, whence

$$P = n^2\pi^2EI/L^2 \quad , \quad n = 1,2,3 \ldots \qquad [6]$$

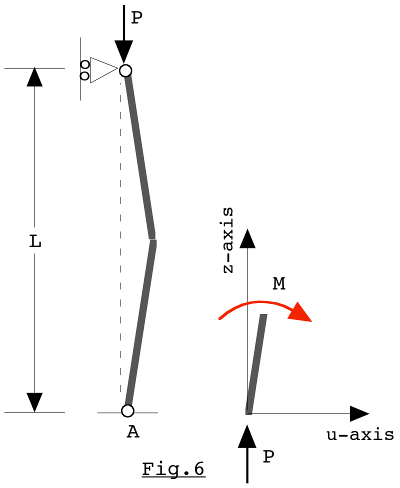



Fig.6

The critical or Euler load (n=1) is $P_E = \pi^2EI/L^2$

Values of n > 1 lead to higher buckling modes i.e. they yield higher values of P. In order to achieve these higher modes sufficiently stiff bracing must be introduced into the system. The first three buckling modes are given in Fig. 7.

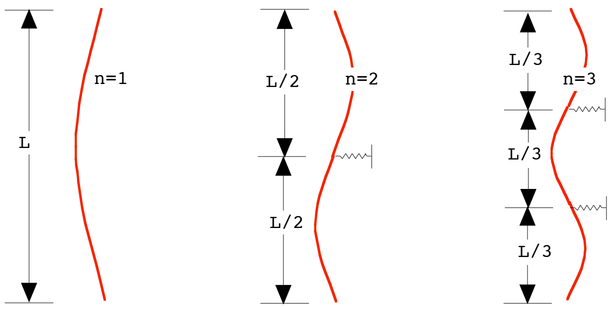



Fig.7

The average stress at elastic buckling is $\sigma_E = P_E/A$

$$= \frac{\pi^2 E}{(L/r)^2} \quad , \; I = Ar^2 \text{ where } r = \text{radius of gyration}$$

The ratio *L/r* is referred to as the *slenderness ratio*. Denote by $\sigma_y$ the yield stress of the material then

$$\sqrt{\frac{\sigma_y}{\sigma_E}} = \frac{L}{r}\sqrt{\frac{\sigma_y}{\pi^2 E}} \qquad [7]$$

Steel stanchions

Steel stanchions will in general have a major and a minor axis for bending. The Euler formula is applied with I being evaluated for the minor axis i.e. the stacnhion will buckle about its minor axis.

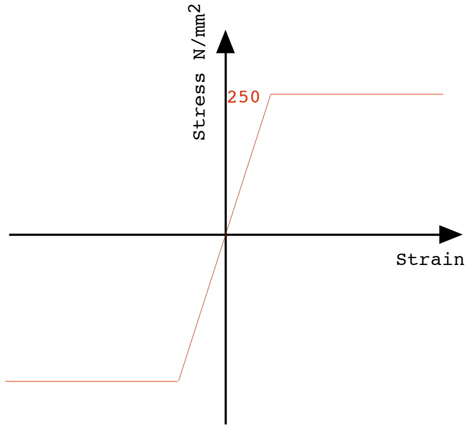



Fig.8 Ideal elastic-plastic stress-strain curve for mild steel

For mild steel the yield stress is 250 N/mm$^2$ and the modulus of elasticity is 205 N/mm$^2$. Noting that $\sigma_y/\sigma_E \geq 1$ then applying equation [7] we get

$$\frac{L}{r}\sqrt{\frac{\sigma_y}{\pi^2 E}} \geq 1$$

or

$$\frac{L}{r} \geq 90$$

In other words ideal stanchions do not fail by buckling for slenderness ratios less than 90. These stanchions are described as *short*, otherwise they are termed *slender*.

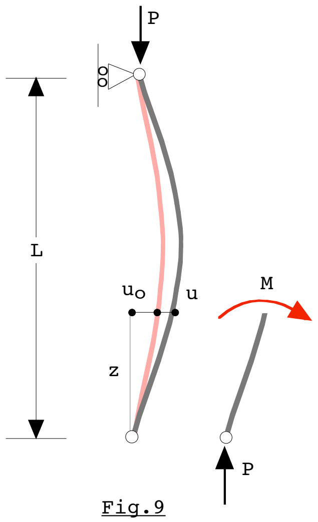



Fig.9

Initial curvature

Let us assume that the column has an initial bow curvature given by $u_o = \delta_o \sin(\pi z/L)$. For equilibrium, see Fig.9

$$M + P(u + u_o) = 0 \qquad [8]$$

Therefore

$$EI\frac{d^2u}{dz^2} + Pu = -Pu_o \qquad [9]$$

or

$$\frac{d^2u}{dz^2} + k^2 u = -k^2 u_o \qquad [10]$$

The CF is of the form A sin(kz) + B cos(kz) and applying the b.c's u = 0 @ z = 0 , L we get A = B = 0.

The PI is of the form $\delta \sin(\pi z/L)$ and substituting in equation [10] we obtain

$$-\{EI\,(\pi/L)^2 + P\} + P\,\delta_o = 0$$

or

$$\frac{\delta}{\delta_o} = \frac{P}{EI(\pi/L)^2 - P} = \frac{P/P_E}{1 - P/P_E} \qquad [11]$$

Equation [11] may be re-arranged in the form

$$\frac{\delta}{P} = \frac{\delta}{P_E} + \frac{\delta_o}{P_E}$$

which forms the basis of the Southwell (experimental) plot.

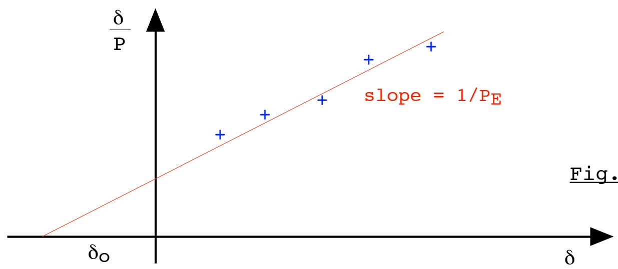



Fig.10 Southwell diagram

## Eccentrically loaded struts - Secant Formula

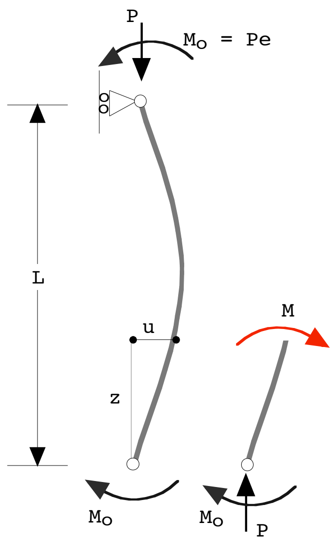



Fig.11

Consider the eccentrically loaded column that is shown in Fig.11. For equilibrium we have

$$M_o + M + Pu = 0 \qquad [12]$$

Therefore

$$EI\frac{d^2u}{dz^2} + P(e + u) = 0 \qquad [13]$$

or

$$\frac{d^2v}{dz^2} + k^2v = 0 \qquad [14]$$

where

$$v = u + e \quad \text{and} \quad k^2 = P/EI$$

The solution of equation [14] is of the form

$$u + e = B\sin(kz) + C\cos(kz) \qquad [15]$$

Boundary conditions:

@ $z = 0$ , $u = 0$ and so $C = e$

@ $z = L/2$ , $du/dz = 0$

Now $\frac{du}{dz} = kB\cos(kz) - kC\sin(kz)$ and so $C = e\tan(kL/2)$

Substituting for A and B in equation [15] yields

$$u = e[\tan(kL/2)\sin(kz) + \cos(kz) - 1] \qquad [16]$$

$$@\ z = L/2 \quad , \quad u = e[\sec(kL/2) - 1] \qquad [17]$$

Let A be the cross-sectional area of the column, $y_c$ the distance from the neutral axis to the extreme compression fibre, I the second moment of area about the neutral axis then

$$I = A\,r^2 = Z\,y_c$$

where r is the radius of gyration and Z is the section modulus in compression. We may now write

Maximum compressive stress

$$= \frac{P}{A} + \frac{Pe}{Z}\sec(kL/2)$$

$$= \frac{P}{A}\left[1 + \frac{ey_c}{r^2}\sec(kL/2)\right]$$

$$= \frac{P}{A}\left[1 + \frac{ey_c}{r^2}\sec\frac{L}{r}\sqrt{\frac{P}{4EA}}\right]$$ the SECANT FORMULA for eccentrically loaded columns.

Consider now the Maclaurin series for sec(x) :

$$\sec(x) = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \ldots\ldots..$$

Noting that $(kL)^2 = \pi^2$ we get

$$\sec(kL/2) - 1 = \frac{\pi^2}{8}\left(\frac{P}{P_E}\right)\left[1 + 1.028\left(\frac{P}{P_E}\right) + 1.032\left(\frac{P}{P_E}\right)^2 + \ldots\right]$$

$$= \frac{\pi^2}{8}\left(\frac{P}{P_E}\right)\left[1 - \frac{P}{P_E}\right]^{-1}$$ approximately, and so the central deflection is equal to

$$\frac{aP/P_E}{1 - P/P_E} \qquad \text{where } a = \frac{\pi^2}{8}e.$$

## Stresses in real stanchions - BS5950

BS5950 Structural use of steelwork in building uses the Perry-Robertson formula to determine the maximum compressive strength that a stanchion can sustain. It is assumed that the maximum deflection is of the form $b/(1 - P/P_E)$. Then the maximum compressive stress is

$\sigma_{max} = P/A + Pb/[Z(1 - P/P_E)]$ , where A = cross-sectional area and Z is the section modulus.

We may therefore write

$$\sigma_y = \sigma + \eta\, \sigma/(1 - \sigma/\sigma_E)$$

where $\sigma = P/A$

$\sigma_E = P_E/A$

$h = b/Z$

This equation may be written as

$$(\sigma_E - \sigma)(\sigma_y - \sigma) = \eta\, \sigma_E\, \sigma$$

The smaller root of this equation gives the required value

$$\sigma = \frac{\sigma_E(1 + \eta) + \sigma_y}{2} - \sqrt{\left[\frac{\sigma_E(1 + \eta) + \sigma_y}{2}\right]^2 - \sigma_E\, \sigma_y}$$

$\eta = 0.001\, a\, (\lambda - \lambda_o) \geq 0$ is the Perry factor

a is the Robertson constant (see Appendix C of BS5950)

$\lambda$ is the slenderness ratio

$$\lambda_o = 0.2\left[\frac{\pi^2 E}{\sigma_y}\right]^{0.5}$$

## Other end conditions

The Euler formula has been derived for a pin-ended column but it can be readily generalised for other end conditions by introducing the concept of an effective length. The effective length $L_e$ is related to the actual length L by $L_e = \alpha L$. The critical load is now $P_E = \pi^2 EI/(\alpha^2 L^2)$. The effective length for various end conditions is given in Fig.12 below

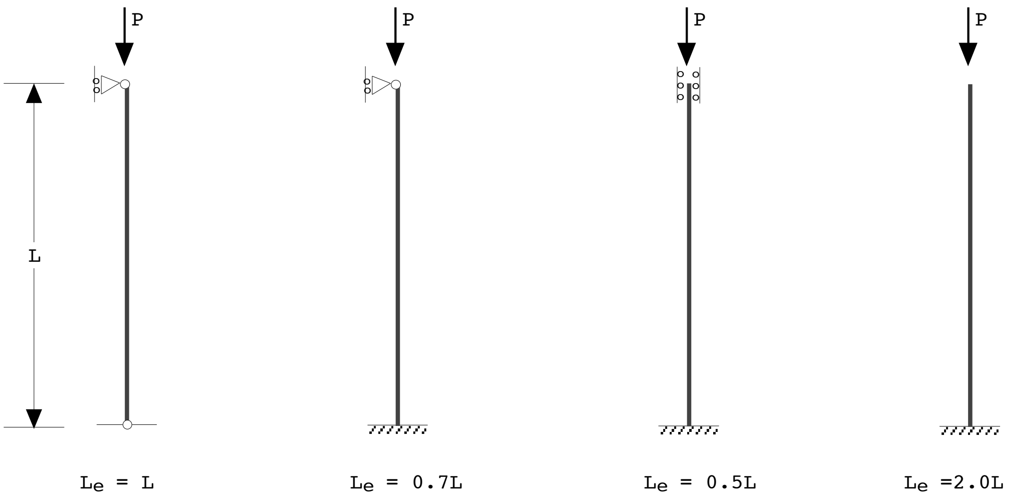



Fig.12 Effective length for various end conditions

Principal Axes

Denote by x- and y- a pair of Cartesian axes. Let u- and v- denote another such pair, see Fig.13. The co-ordinates of point P are related to each other in the following way:

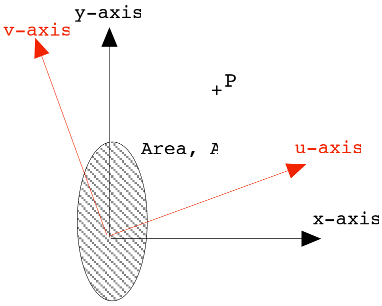


Fig.13

$$u = \cos\theta \ x + \sin\theta \ y$$
$$v = -\sin\theta \ x + \cos\theta \ y$$

The second moments of area of the cross-section shown in the figure are:

$$I_x = \int_A y^2 \, dA \quad , \quad I_y = \int_A x^2 \, dA$$

The product moment of area is

$$I_{xy} = \int_A xy \, dA$$

These quantities may be written w.r.t the u-, v-axes as follows:

$$I_u = \int_A v^2 \, dA = \int_A (-\sin\theta \ x + \cos\theta \ y)^2 \, dA = \cos^2\theta \ I_x + \sin^2\theta \ I_y - 2\sin\theta \cos\theta \ I_{xy} \qquad [A1]$$

$$I_v = \int_A u^2 \, dA = \int_A (\cos\theta \ x + \sin\theta \ y)^2 \, dA = \sin^2\theta \ I_x + \cos^2\theta \ I_y + 2\sin\theta \cos\theta \ I_{xy} \qquad [A2]$$

$$I_{uv} = \int_A uv \, dA = \int_A (-\sin\theta \ x + \cos\theta \ y)(\cos\theta \ x + \sin\theta \ y) \, dA$$

$$= \sin\theta \cos\theta \, I_x - \sin\theta \cos\theta \, I_y + (\cos^2\theta - \sin^2\theta) \, I_{xy}$$

$$= \frac{1}{2} \sin 2\theta \ (I_x - I_y) + \cos 2\theta \ I_{xy} \qquad [A3]$$

For $I_u$ to be a minimum or maximum then $dI_u/d\theta = 0$. Hence by differentiating [A1] we get

$I_{uv} = 0$ for principal axes i.e minor and major axes.

Equation [A3] gives

$$\tan 2\theta = \frac{2 I_{xy}}{I_y - I_x} \qquad [A4]$$